MATHEMATICS AND THE PHYSICAL WORLD

Inaugural Lecture of the
Professor of Applied Mathematics
delivered at the College
on March 2, 1954
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MATHEMATICS can be described as a study of abstractions and idealizations. The concept of a *number* is abstracted from experience in the following way. If we have a stone, and we are then given another stone, we have something we recognize as independent of when or where this happens—what we have learnt to describe as *two stones*; and if we have a stick and are then given another stick we always have what we call *two sticks*. It is a matter of experience that there is something common to the similar experiments with objects of different kinds. And from this the mathematician has abstracted the idea of a number *two*, and an operation of addition (one plus one equals two), without reference to stones, or sticks, or any other objects. A good deal of thought has been given to the development of a self-consistent theory of number based on the simplest possible abstract postulates, initially inspired by experiences such as I have described.

The straight line with no thickness, intersecting a second straight line in a point of no size, is another abstraction from experience, which we meet in Euclidean geometry. An object with a finite mass but no size—a *point mass*—is a useful abstract concept, an idealization of a real particle of matter, which is introduced in mechanics to permit greater precision in the logical development of the theory of how material objects move about when subjected to forces.

The purpose of such processes of abstraction and idealization is always to make possible rigorously logical deductions about numbers, or points and lines, or particles of matter. The need for conciseness in expression, as well as for preciseness of concept, has been felt more and more as the subject has developed, and specialized notations have arisen, quite naturally at every stage, to such an extent that a modern mathematical paper often appears quite incomprehensible to anyone who is not a mathematical specialist. The advantages of a compact notation are unmistakable if you think how much easier it is to write, in symbols, $\sqrt{(x^2+1)}$ than it is to say, every time it occurs: the number obtained from the number you first thought of by multiplying it by itself, adding one to the product, and then working out the number which, when multiplied by itself, will give that sum. Mathematics, then, has been compelled to develop its own language, which is very difficult to translate into ordinary English; and which is essentially a written language, as it is in danger of losing some of its preciseness if conveyed by speech alone. The language of mathematics can express in a page of well-chosen words and symbols a sequence of ideas which it would take many pages to describe in ordinary words alone. It is a language with a beauty of its own; it can convey (from one mathematician to another) an elegance in a gradual unfolding of ideas which would be lost in translation into any other medium.

The habitual use by the mathematician of a special language, and that a written language, makes it difficult for him to talk about his work in any detail to an audience who are not specialists in his own field. And yet the great French mathematician of the eighteenth century, Lagrange, maintained that a mathematician has not thoroughly understood his own work unless he is able to explain its significance effectively to the first man he meets in the street. Before I submit to something in the nature of a Lagrangian test, I should like to say a little more about mathematics in general, and about applied mathematics in particular. The subject of Applied Mathematics has recently received new recognition in this College in the creation of the Chair of which I have the honour to be the first occupant. It is more important, I feel, on this occasion, that I should try to convey to you something of the outlook of an applied mathematician, his approach to mathematics and to the scientific description of the real world, and what he can hope to achieve by the use of his special skills, than that I should speak exclusively about the particular physical problems which have been my especial interest. I shall mention such specific problems, but not for their intrinsic interest alone; rather, because I can best illustrate in that way the attitude a mathematician must adopt in treating problems which arise in the physical world, and demonstrate the significance which can be attached to the results of his research.

It is implicit in these remarks that there is more than one kind of mathematician, classified as *pure* or *applied* according to whether his interest is in abstractions for their own sake, or in mathematical idealizations in relation to the physical experiments which give rise to them; within these broad categories, there are represented a wide range of different philosophies. The primary classification is of men, by their attitude towards the physical world, rather than of mathematical systems, by their content.

We have first what we might call the extremely pure mathematician, who is fascinated by his subject as a creative art and requires no other justification for pursuing his life's work. He is concerned entirely with the intrinsic beauty of his theorems, and the significance of his results in relation to other purely mathematical ideas. He neither takes inspiration from the physical world, nor is he interested in the significance of his discoveries in relation to the physical world; and he may even glory in their uselessness, like the extremely pure mathematician in the story, who shot himself when he learned that one of his beautiful theorems in four-dimensional geometry was being used in a study of the chemistry of brewing.

The late Professor G. H. Hardy, a founder of the modern Cambridge school of rigorous analysis and undoubtedly one of the great mathematicians of this century, in his book *A Mathematician's Apology* justifies his life as a mathematician by his having created something worth creating in mathematics, having 'added something to knowledge, and helped others to add more'. He does not subscribe to the view that uselessness is itself meritorious, but it is significant that he thinks it worth while to add:

> I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. I have helped to train other mathematicians, but mathematicians of the same kind as myself, and their work has been, so far at any rate as I have helped them to it, as useless as my own.

The thesis that all profound mathematical thought must be completely isolated from any real associations is, I wish to emphasize, a purism. It is, however, sometimes put forward as the only view any self-respecting mathematician could possibly hold, and this has contributed to a common misunderstanding about the nature of applied mathematics, which I shall return to in a moment.

Although there are a few mathematicians who appear to have lost sight of the origin of mathematical thought in abstraction from experience, there are many more, whose interest still lies in methods of analysis rather than in physical science, whose greatest work has been the resolution of some purely mathematical difficulties arising as

a result of the study of natural phenomena. Fourier's study of the properties of the trigonometric series which are named after him was inspired by their natural occurrence in the exact description of the flow of heat from the hotter to the colder parts of a solid body. 'The profound study of nature', he declared, 'is the most fruitful source of mathematical discoveries.' Henri Poincaré endorsed this view, and his researches on a very wide range of mathematical topics lend considerable support to it. Poincaré, who died in 1912 at the height of his creative period, has been described as the last universalist, for he was the last great mathematician who took practically the whole of mathematics within his province and produced an immense amount of new mathematics of all kinds. His early work on differential equations, equations which arise in the analysis of physical situations, is characteristic of one who follows Fourier's philosophy.

We have yet to consider the mathematician who takes his main inspiration from the observation of natural phenomena—from physical, chemical, and occasionally biological science—and who seeks to draw conclusions from the results of one experiment regarding what will happen in another, for whom the mathematical method is important only in so far as it is powerful in achieving this end; that is the applied mathematician. A real physical situation often needs drastic simplification before it can be represented by means of mathematical relationships between simply defined physical quantities; but the equations which the applied mathematician constructs as a basis for logical extension, however much they are idealized, are essentially physically significant equations. If he is able to deduce certain consequences from his formulation of a natural problem in a set of equations, he is mainly interested in those which are capable of physical interpretation.

Poincaré's ability to contribute work of the highest quality to all the known branches, both of pure analysis and of theoretical physics, is not likely to be seen again. For mathematics has expanded, and is expanding, to such a degree that no one man could now have a comprehensive knowledge of all modern developments, or make a significant original contribution to more than two or three main fields of research. The need for specialization does not present any great problem to the pure mathematician, for he has no objection to working in a restricted field, such as the geometry of spaces of many dimensions, where his own contribution will be related only to the work of other geometers—no matter how far-reaching are modern developments in the theory of differential equations or astronomy. But the impossibility of fully comprehending the whole range of mathematical thought does, to some extent, affect the outlook of the applied mathematician.

He could not now, even if he wished to, claim to be an expert in all known mathematical techniques, ready to use any of them at a moment's notice, to solve whatever problem the experimentalist might bring along to him. Yet the term *applied mathematician* is often, quite erroneously, taken to mean a technician of that kind, a tame mathematician who can solve any awkward differential equation that arises in the interpretation of experiments, or even work out the results if the calculations become too heavy. I must do my best to dispel this frequent misrepresentation of the nature of applied mathematics.

It is true that all mathematical developments, however they may have arisen, must be regarded as potentially useful as a means of describing natural phenomena. An applied mathematician may find that someone else has developed—either as a piece of pure mathematics or as

part of a physical theory—a mathematical structure which is ideally suited to the description of some natural process. For example, Einstein found in Ricci's absolute differential calculus—or the tensor calculus, developed purely as differential geometry—exactly what he needed for the formulation of the theory of general relativity. On the other hand it may be necessary, not simply to adapt someone else's mathematics, but to create an entirely new mathematical structure in order to represent the essential features of a newly observed physical phenomenon. Now it seems to be part of the misunderstanding about the scope of the subject, closely related to the purist's view that mathematics-for-its-own-sake is all that matters, that there could be no artistic value in mathematics inspired by such mundane things as the motion of the planets, the flow of a glacier, the northern lights, or conditions inside the sun; no beauty in mathematics which leads to a better understanding of the structure of a molecule, to greater facility in forecasting the weather, or to the development of a new process of textile spinning. It would indeed be surprising if the inspiration for creative work in any medium could come only from the work of other artists, and from no external source; in mathematics, at any rate, it is quite false to suppose that this is so. The creation of new physical theory can cause great emotional satisfaction to the author, and its perfected formulation is worthy of being recorded in the literature to be read with pleasure by the future generations of mathematicians, irrespective of its immediate usefulness. Professor C. A. Coulson, in his inaugural lecture in Oxford under the title *The Spirit of Applied Mathematics*, epitomized his subject in a single sentence:

> Applied mathematics is an intellectual adventure in which are combined creative imagination and authentic canons of

beauty and fitness; they combine to give insight into the nature of the world of which we ourselves, and our minds, are part.

The name *applied mathematics* is no doubt itself responsible for some misunderstanding, and it is to be regretted that the accepted synonym *natural philosophy* is not more widely used; it is a subject where mathematical methods are subsidiary to scientific implications.

An essential property of the language of mathematics is that it is self-consistent. In pure mathematics, the fundamental postulates—the universally accepted premisses from which logical deductions are made—are so well defined that there is always a unique correct answer to any problem, as in simple arithmetic, no matter what method of procedure has been adopted in the proof. For example, in Euclidean geometry, it is readily proved that the sum of the three angles of any triangle is equal to two right angles; and if, in some problem in Euclidean geometry we evaluate the angles of a triangle and find that they add up to more, or less, than two right angles, then we have made a mistake. We have no doubts about it.

On the other hand, let us examine what conclusions are to be drawn if we arrive at an unexpected result in a problem in applied mathematics; an example will make the situation clear. By means of the Newtonian theory of gravitation, based on the assumption of a simple natural law that bodies are attracted to each other by a force directly proportional to each of their masses and inversely proportional to the square of the distance between them, the positions of the planets at future times may be calculated from their known positions now and in the past. According to this theory, the planets move round the sun in orbits which are approximately ellipses, and which would be exactly ellipses if it were not for the small disturbing influence of one planet on the motion of

another. If there is disagreement between the predicted and observed motion of a planet, the cause of the discrepancy is not immediately obvious. We may conclude that either

(i) there has been an error of observation, or
(ii) we have made a mistake in interpreting the theory, perhaps by an error in algebra or arithmetic, but perhaps through ignorance of the existence of all the other planets whose disturbing influence should have been taken into account, or
(iii) the fundamental assumptions on which the theory is based require modification in the real solar system.

We must in fact be prepared to relinquish the theory, as one providing an inadequate approximation to the truth, if we cannot otherwise resolve the difficulty.

There are two famous instances where precisely the disagreement I have outlined has arisen in the study of planetary motion, when the possibilities of observational error of the necessary order of magnitude and of mistakes in calculation could be decisively ruled out. In the first case it was found that a slight irregularity in the motion of the distant planet Uranus could be accounted for if a hitherto unobserved planet were present in a more distant orbit. Such a planet, afterwards called Neptune, was observed in 1846 in just the position predicted by Leverrier in France and by Adams in this country. Newton's theory of gravitation, together with the elaborate astronomical predictions based on it, was left unchallenged as an accurate description of the motion of real bodies.

Later, Leverrier made a similar attempt to explain an anomaly in the motion of Mercury, the planet moving comparatively rapidly in an elongated orbit close to the sun with a period of revolution about the sun of only eighty-eight days. The calculated perturbations due to the other planets explained an angular advance in the position at which the planet came nearest to the sun at the rate of 8′ 52″ per century; the advance observed was at the rate of 9′ 34″. The name Vulcan was given to a new planet moving in an orbit even nearer to the sun than Mercury, whose presence would explain the excess of about 42 seconds of arc which could not be accounted for by errors of observation or calculation. But no such planet has ever been observed. When Einstein's general theory of relativity provided an alternative explanation of the motion of Mercury, without invoking the existence of an unobservable planet, it was realized that Newton's theory of gravitation, as an exact theory for the faster-moving bodies at any rate, must be abandoned in its favour. The two theories agree, within the limits of observational error, so far as the rest of the solar system is concerned.

In such a way as this, an applied mathematical development of some basic physical hypothesis may change its status, as theoretical physics, overnight. It may continue to provide a rough approximation to, although it can no longer rank as a completely satisfactory explanation of, the natural phenomena to which it relates; in other circumstances it may have to be regarded as altogether untenable, as theoretical physics, because of conflicting experimental evidence. When this happens, the mathematician may not feel obliged to abandon completely his research along the lines discredited by the physicist. In exploring theoretical simplifications or generalizations of what is currently held to represent physical reality, he is at liberty to allow his imagination freer rein than the physicist, who must pay due regard to the details of what he actually observes in his experiments.

Mutually incompatible alternative theories are often developed simultaneously, and it is not always clear, until several possible consequences have been investigated experimentally, which analysis of the situation is the most realistic. It is well known that Newton built up a corpuscular emission theory of light—the theory of light rays regarded as the paths of definite particles of light—and that this was later rejected by physicists in favour of the wave theory associated with the name of Huygens, as a result of crucial experiments on optical interference. But in some of his work, Newton considered light to be a wave phenomenon, and showed a characteristic caution in giving no clear indication which theory he considered really to represent the truth. The two apparently incompatible theories have been pursued simultaneously, notwithstanding that one or other of them has from time to time been out of fashion as theoretical physics; this independent, imaginative attitude has had its reward in the integration of the two points of view in Einstein's relativistic light-quantum theory, an essential constituent of the modern wave-mechanics of matter and radiation.

I should like to illustrate the procedure which is likely to be followed in a mathematical approach to a new physical situation by discussing some general questions which I have myself been occupied with; I refer to the study of the deformation or flow of a solid or a liquid when it is subjected to forces tending to change its shape.

Many of the materials of industrial importance today are new materials—new alloys, plastics of all kinds, synthetic fibres for textiles—whose properties are different from those of the mainly natural products which have, until recently, been almost exclusively used for all our needs. It is of immense practical importance that we should be able to describe the mechanical response to stress of these unfamiliar materials in precise terms; such a description can help us to make an advance assessment of their potential new uses. The suitability of the old, familiar materials, for the purposes for which they are habitually used, has become established as a result of experience over a long period: stone and brick for building houses, rubber for electrical insulation. The ultimate test for a new material is a practical trial preceded by laboratory experiments.

In the absence of any comprehensive theory of the mechanical response of solids and liquids to applied stresses, laboratory experiments have often been of a purely empirical nature, often following an industrial or laboratory practice which has been found appropriate for a different material with simpler properties. If industrial tests have by custom been purely subjective, and the properties usually assessed are somewhat ill-defined, there is no sound basis for laboratory tests at all. This difficulty is acute, for example, in the cheese industry. The mechanical properties of cheese are very important commercially, and the standard method of testing a cheese for ripeness is a partly subjective assessment of them: the cheese-maker observes the impression made by pushing his thumb into the cheese, and at the same time infers from the feel of it whether the cheese is ready for eating. This subjective assessment is more reliable than any single laboratory test that has yet been devised to replace it; it is not known exactly what mechanical property can be correlated with the experienced craftsman's judgment, or even what all the complicated mechanical properties of cheese actually are.

There are one or two simple types of mechanical behaviour which have been exhaustively studied. A bar of metal under tension extends by a very small proportion

of its length: a pull of 10 tons weight per square inch of cross-section produces in steel an elongation of less than one-tenth of 1 per cent. A far-reaching mathematical theory of behaviour of this type has been built up during the past three centuries, based on the simplifying assumption, proposed as an idealization from experiment in 1678 by Robert Hooke, that all extensions, or *strains*, are strictly proportional to the loads, or stresses, which produce them. The theory has provided a close approximation to the behaviour of many real materials; many important constructional materials obey Hooke's law of proportionality, and the engineer can use this classical elasticity theory to calculate the safe load for a steel girder or a masonry dam. But it is easy to see that it will not provide a very good approximation for some other solids. For instance, when rubber is loaded, the response to the stress is of a different order of magnitude altogether; a tension of 100 pounds weight per square inch of cross-section can stretch a specimen to three or four times its original length. A simple experiment also shows that the extension is not simply proportional to the load.

Another idealization of mechanical behaviour, which provides a very close approximation to that of ordinary mobile liquids like water and mercury is embodied in what is known as Newton's viscosity law. The basic assumption is that if one had two large horizontal flat plates, close together with liquid in between, the force required to pull the plates apart, sideways, parallel to themselves, would be in direct proportion to the relative speed of pulling—due to internal friction, or *viscosity*, in the liquid. A mathematical theory of the behaviour of this kind of liquid when flowing in different ways has been built up gradually since the time of Newton; it accounts satisfactorily for the flow properties of honey, glycerine, thick oils, and many other viscous liquids. But the flow properties of colloidal suspensions of small particles in a liquid, and solutions obtained by dissolving in a liquid a solid known to have very large molecules, are not accounted for by the classical theory. Long molecules in the form of a chain of many simple molecules united chemically—linear polymers as they are called—are present in natural materials such as rubber and cellulose from plants and proteins in animal products, and are now common in the synthesized materials known collectively as plastics. The increasing commercial use of these, sometimes in solution, has attracted attention to the existence of unexpected mechanical properties of many kinds.

If we confine attention to materials which do not appreciably alter their size if they are subjected to reasonably small forces (that is, if we exclude gaseous materials), all these can conveniently be divided into two mutually exclusive categories: we call them *solids* if they do not change their shape continually when subjected to sufficiently small stresses; and *liquids* if they do change their shape continually when subjected to forces, however small, maintained for a finite time.[1] Broadly speaking, the work done in deforming a solid is mostly stored as internal potential energy, available to make the solid spring back into its initial configuration at a later time; the work done in deforming a liquid is mostly done against internal friction and dissipated as heat. In the majority of real materials, however, a certain amount of elastic potential energy is stored in the material, either transiently or semi-permanently, and a certain amount of the mechanical work, needed to change the shape of the body, is lost in the form of heat due to internal friction.

[1] The rheological definitions of solid and liquid differ from those found convenient in physical chemistry.

In general, a material chosen at random will have some of the properties ordinarily associated with solids, and some of the properties of simple liquids; and when it does not approximate to either of the simple extreme cases, it is then that its properties can prove particularly difficult to formulate with any precision.

Sealing wax responds to stresses applied quickly just as a hard solid, even to the extent of showing brittle fracture, but it must be classed as a liquid because it flows, with a permanent change of shape, even under its own weight. Certain jellies have the appearance of a solid, rather like a table jelly, and are undoubtedly to be classed as solids by the definition; yet if they are disturbed by a momentary shaking they may show for some seconds, or minutes, or hours, all the appearance and properties of a mobile liquid; they are then said to be in the *sol* form. On being left to stand the material returns to the solid, or *gel*, form. Such a reversible sol-gel transformation has been given a special name—*thixotropy*. There appears to have been at one time a tendency to pronounce such magic words as 'Thixotropy!', 'Spinnbarkeit!', or 'Rheopexy!' when any quite unaccountable mode of mechanical behaviour was observed in the laboratory, but they were of no avail in opening the door to an understanding of the phenomena, and appear to have gone out of fashion.

The elasticity of sealing wax is different only in degree from the elasticity readily observable in certain dilute polymer solutions: these have the appearance of ordinary mobile liquids, but if air bubbles are present when a bottle of the liquid is jerked, the bubbles are seen to oscillate rather as they would in a disturbed jelly. Intermediate in degree between the elasticity of sealing wax and of a dilute polymer solution, there are liquids which are mobile enough for any surface irregularities to be smoothed out by flow under the liquid's own weight within a few hours of being put in a beaker, and yet elastic to the extent that a drop of the liquid can be bounced on the floor like a rubber ball, with no visible flattening of the surface after it has struck the floor. The liquid has a very different response to slow and to rapid movements. A characteristic feature of the behaviour of liquids with elasticity is that when they are stirred with a rod they tend to climb up the rod rather than to be thrown off it by centrifugal force as water is. Any animal which finds itself in a shallow pool of liquid of this kind can very soon become hopelessly entangled if it moves its limbs rapidly in an effort to get out, and such liquids are sold commercially for catching mice.

All these rather remarkable phenomena are food for serious thought for the mathematician. There is a clear need for a comprehensive theoretical discussion of mechanical response to stresses, which must include, within its scope, all the types of behaviour which are known to be possible. I shall now speak of the general problems which are involved in this requirement, from the mathematician's point of view, with some indication of how far they have been resolved.

The first general problem is that of relating the properties of a small sample of a solid or liquid with the properties of its constituent molecules. In the case of a crystalline solid, such as a metal, where the atoms are known to be in an ordered array, it is possible to relate the elastic constants for a single crystal with the lattice spacings; and the occurrence of slow plastic flow, resulting in permanent deformation when the material yields under a critical load, can be explained in terms of dislocations in the array of atoms, each atom moving spontaneously so as to reduce slightly the internal potential energy of the crystal. But in the case of amorphous solids

and liquids, where there is no long-range order among the constituent molecules, and in which there occur continual changes in the relative positions of a molecule and its neighbours, there is as yet no completely satisfactory statistical-mechanical theory to relate molecular and macroscopic properties. There is, however, a notable exception: a structural theory of rubber and rubber-like materials, which explains the observed mechanical properties of vulcanized rubber with some accuracy, has been based on the following idealized picture of the molecular arrangement in rubber.

The rubber molecule is to be envisaged as a long-chain polymer in which the constituent small molecules have freedom of relative movement, subject to certain restrictions, rather as the links of an ordinary chain can move, so as to give the molecular chain some flexibility. We must think of the chain in a continual state of agitation, of a degree depending on the temperature. In the process of vulcanization of the rubber one or two points of each chain are chemically bonded to points of neighbouring molecular chains, and the free lengths of all the chains between junction points with other chains are of all possible shapes and sizes. The mathematical theory of rubber elasticity is a statistical theory, dealing with the free lengths of chain in a small finite piece of rubber, grouped according to their configurations at any instant. The theory explains not only the magnitude of the stretch of a piece of rubber, and its non-linear variation with the load, but also why, when we suddenly stretch a piece of rubber, it feels noticeably warmer, and when we suddenly release a specimen from the stretched position it feels, for a moment, cooler than its surroundings. The idealized molecular picture on which the statistical theory has been based is admittedly not a complete picture of all that is known about the constitution of rubber, but the main features of the molecular arrangements are taken into account and the main characteristics of the macroscopic rubber are deduced from them. The selection of what seem to be the essential features of a complicated physical situation is always required of the applied mathematician; without such selection he would usually find his real problems were represented by equations so complex that he would not be in a position to draw conclusions from them.

A less ambitious structural theory of the flow of liquids, relating macroscopic properties with those of microscopic constituents, in simple cases, instead of directly with molecular arrangements, has provided a partial explanation of certain non-Newtonian flow characteristics of real viscous liquids. The increase in viscosity when small solid particles are present in suspension in a liquid was first calculated by Einstein at the beginning of this century. More recently an explanation has been given, in terms of the elastic energy of suspended colloidal particles in a two-component system, of such effects as the tendency of elastic liquids to climb up the rod which stirs them. The suspended particles may be microscopic elastic solid particles (for example, finely divided rubber), or minute droplets of one of the two liquid components of an emulsion, suspended without dissolving in the other (liquid) component. If we think of the long, polymer molecules of a solid *dissolved* in a liquid, which may be coiled up rather like an irregular spiral spring entangled within its own coils, as having the nature of sub-microscopic elastic particles, then we may think of the microscopic theory of liquid elasticity as providing the beginnings of a molecular theory for polymer solutions. Certainly, some polymer solutions have the type of mechanical properties predicted for liquids containing elastic particles, although no precise quantitative relationship between theory and experiment has yet been established.

Let us pause for a moment to consider the significance of all this. In these days of synthetic chemicals, it is, to a limited extent, possible to make new substances, particularly polymers, with 'tailor-made' molecules, built up in a prescribed manner from simpler molecules by controlled chemical reactions. An industrial requirement may be for a material which must have, among other things, certain quite complicated mechanical properties in bulk, either as a solid or in solution in a suitable solvent; for example, there may be a demand for a new substitute for rubber, or a substitute for wool. Without some correlation —however rudimentary—between molecular properties and macroscopic mechanical properties, such requirements could only be satisfied fortuitously; with a complete correlation it would be possible in principle to fulfil a specific requirement as a result of a systematic series of trials.

The second general problem for the mathematician is the study of the deformation and flow of a solid or liquid regarded as sufficiently represented by a homogeneous, continuous material, that is the relation between the behaviour of the material in bulk with that known (from simple experiments or structural theory) to be characteristic of a single macroscopic element of the material. Theoretical results which may ultimately be of use to the engineer—who may, for example, be concerned to know whether new materials will be potentially useful in certain manufacturing processes—must deal with flow under prescribed external conditions such as those of the rolling mill or the extrusion jet.

To take a simple example, it may be of interest to calculate the pressure required to pump a thick paste or slurry through a pipe. The question might first have arisen as one aspect of an engineering problem: it may be necessary to know ultimately whether it will be more economical to pump newly mixed concrete continuously through pipes, or to move it in batches by truck, in planning some large-scale constructional work. The engineer may require only a single numerical answer, probably only an approximate one, to reach his decision in a particular case. But the flow problem alone may require an elaborate theoretical analysis if the material to be pumped has physical properties which are not particularly simple. In this case the results are likely to be of general interest, and the mathematician embarks on a determination of the power required to force a material with both elastic and viscous properties at a constant rate through—in the first instance—a uniform, straight, horizontal pipe with a circular section. If he succeeds, he may be able to go farther, and describe what happens when the pipe is not uniform, not straight, not horizontal, and possibly not even circular in section.

Some of these modifications may prove too complex for a general analysis. In that case, if the need arises in a particular instance, it may be necessary to resort to (perhaps laborious) numerical calculations, to be done by a team of computers with calculating machines. A purely numerical calculation makes use of the measured dimensions of the equipment and the measured constants of a particular material at the outset, and gives a numerical result which would in general be quite irrelevant if the equipment or the material were changed. The result is of no general interest. The mathematician will not be content until he has succeeded in deriving a general result for any equipment of a certain type, and any material of a certain type, so that the answer to any specific problem can be deduced from it by substituting actual sizes and material constants in the general formulae.

Let us see how the mathematician can hope to describe the flow in bulk of a new type of material. The kind of material he has in mind may have been observed to have certain properties in the laboratory, or it may be an idealized liquid corresponding to certain arrangements of molecules or microscopic particles, with macroscopic properties obtained by inference from the properties of its constituents. He must first formulate these properties in a set of equations: the *equations of state* of the material. These must relate geometrical quantities defining the shape of a typical small piece of the material, the forces acting upon it, and the temperature, all considered to be varying with the time. The equations are analogous to the well known equation of state for an ideal gas: the volume multiplied by the pressure is a constant times the absolute temperature; but they are usually derived as a set of six or more differential equations, instead of a single algebraic equation. They must not only represent the properties which have been observed or calculated for an arbitrary small portion of the material, under the conditions appropriate to the first observations or the calculations, but must also define the properties under all possible conditions—all types of deformation and all types of applied forces—consistently with the known properties.

Even this is not all we require of these equations: they must be so constructed that they really do represent physical properties of the material, and nothing else. It is very easy, as it happens, to write down equations which define the behaviour of a material under all conditions, but which depend on the physical properties of the material and also on the set of symbols the mathematician has chosen—his frame of reference. This means the equations are not self-consistent physically (although they may satisfy all the purely mathematical rules for compatibility) and could have no true physical significance. For example, they might be a set of equations proposed for describing the properties of the metal of which a spring balance is made, and yet they may predict a different reading of the spring balance when extended by a pound weight according to whether the balance is at rest or moving with uniform velocity. Such inconsistencies must be avoided, and only certain forms of the equations of state are admissible as having possible physical significance with complete generality.

There we have a difficulty in formulating a physical concept in exact language—the language of mathematics. It is just such processes of formulation which give applied mathematics its distinctive character; they distinguish it from pure mathematics and from physics. There is no routine procedure for constructing the basic equations for a mathematical theory from the results of suitably designed experiments; each new physical situation must be considered on its merits. Completely new mathematics, new types of abstract quantity, and new types of relationship between such abstract quantities, can arise naturally in the process of formulation. Their introduction may complicate the interpretation of the consequences of the original physical ideas; but the obligation to interpret his results in physical language is undoubtedly on the mathematician, in any case, whatever mathematical method he has used. Provided he ultimately fulfils that duty there is no possible disadvantage in the complexity or subtlety of his notation, and he is quite at liberty to extend his special language in whatever way he chooses, for the purpose of examining all the implications of the basic concepts.

There are as yet many types of material for which the

experimental data are too meagre for equations of state to be determined. This deficiency can be remedied by making more physical experiments, carefully designed to provide the missing information; only the mathematician can say precisely what information he requires as a basis for a theory, so he must play a part in the design of the experiments.

With the representation of the available physical data about a solid or liquid in a self-consistent set of equations of state, the main task of the applied mathematician is over. He is in a position to reduce any physical problem concerning the deformation or flow of the material in bulk to a set of differential equations with certain boundary conditions—in fact to a purely mathematical problem.

I do not wish to belittle the difficulty of solving differential equations. There may or may not be known an analytic method of solving the particular equations arrived at in the course of a physical problem. They are more likely than not to be equations which no pure mathematician has thought of inventing for their own sakes; the applied mathematician must then seek an analytic solution himself, perhaps simplifying the conditions of the physical problem he is studying until he can solve the corresponding equations and find an answer which he can translate into the ordinary language of physics.

The possibility of modifying the precise conditions under which a material is considered to be deformed gives the mathematician a good deal of choice in the development of a theory of flow in bulk. If the material is one whose properties are comparatively simple, such as the prototypes of Hooke and of Newton, on which the classical theories of elasticity in solids and of viscosity in liquids have been based, it is possible to select many sets of conditions which lead to tractable differential equations. There are then placed on record a sufficient number of examples of the mode of deformation for the behaviour of the material under somewhat more complicated conditions to be inferred by analogy.

The freedom of choice of boundary conditions, exercised to make the mathematics tractable, is of even more importance when we come to a material whose properties cannot be represented by simple equations at all. Here it may determine not only the line of theoretical development but also the experimental research undertaken on available materials; this is happening in the current work on elasticity in liquids. The mathematician is in a position to say to the physicist: 'If you examine the flow in an apparatus of *this* kind, I shall be able to say whether or not the liquid you are dealing with has the kind of elastic property you suspect, but if you use *that* kind of apparatus, I am afraid I shall not be able to help you in interpreting the results of your experiments because of the complexity of the equations which represent that type of motion.' The mathematician plays a leading part, in fact he must take some of the initiative, in the design and interpretation of experiments in this field of scientific research.

The role which the mathematician can play in research in collaboration with the physicist and the chemist must be borne in mind in relation to the type of mathematical education we offer in schools and universities. The balanced association of mathematical methods and scientific inference, which is the essence of applied mathematics, should be introduced early in the teaching of mathematics if we are to train the student to use his knowledge in the pursuit of scientific truth. He must, of course, become acquainted with the standard methods of pure mathematics, with geometry and analysis; and we can try to

bring it about that he learns from these something of the power and the beauty of mathematics itself. But competence to use purely mathematical techniques will not alone make him a competent applied mathematician. To make full use of his mathematical training he will require also practice in the appreciation of physical situations, in selecting the salient features of a problem—in mechanics, or electricity, or hydromechanics—and expressing their essence in the language of mathematics. His pure mathematics will help him to deduce the mathematical consequences of his equations, but it will not teach him how to interpret his deductions in the language of physics. Without any appreciation of physical phenomena, the mathematician will be unable to take any initiative in the planning and interpretation of the crucial observations of the physical world on which his advice may be sought. What is more regrettable is that he will fail to catch an occasional mathematical inspiration from the wonders of nature which, on analysis, are often so beautiful in their simplicity.

The need for physical insight as well as mathematical technique makes applied mathematics a more difficult subject of study than mathematics-for-its-own sake. The power to translate scientific facts into mathematical equations, and facility in appreciating the physical significance of their mathematical implications are not something the student can acquire quickly if he has been taught only pure mathematics, physics and chemistry, as separate compartments of knowledge.

According to a definition of mathematics given by Bertrand Russell, a mathematician is one who does not know what he is talking about, nor whether what he is saying is true; and, one might add, he doesn't care either. You may be convinced, after hearing this lecture that this is indeed the case. But if, after hearing me, you are prepared to admit the existence of a species of mathematician who, at least cares whether what he is saying is relevant to the real world, then I have achieved something of my purpose.

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