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The Mathematics of Patterns and Arrangements

Inaugural Lecture of the Professor of Pure Mathematics, delivered at the University College of Swansea on 2 December, 1971

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THE MATHEMATICS OF PATTERNS AND ARRANGEMENTS

I

The first Professor of Mathematics in this College was A. R. Richardson, who was appointed Head of the Department of Mathematics in 1920 and held the post until 1940, when he was obliged to retire on account of deteriorating health. He had received severe wounds in the 1914-18 war, and subsequent operations had left him with a legacy of progressive illness. I am the only member of the present Department of Pure Mathematics who knew Richardson, and I would like to take this opportunity to pay tribute to him as the founder of a department which has always had a high reputation as a centre of active research in Mathematics. The research tradition has been most worthily fostered by his successors, but they would have had a harder task if the foundation stone had not been well and truly laid in the nineteen twenties. In many ways Richardson was ahead of his time. He lectured on Abstract Algebra to undergraduate classes when it was certainly not fashionable in the universities of Great Britain. It has now begun to percolate through to A-level, and even O-level, courses in schools. His early academic training was as an engineering student at the Royal College of Science ; I believe he made himself rather unpopular with the staff of the Engineering Department at Swansea by devising an Intermediate course, largely for their students, with seemingly new-fangled, irrelevant and abstract ideas such as Boolean algebra. He was elected a Fellow of the Royal Society in 1946, and continued his researches in Pure Mathematics until his death in 1954.

Because of the research tradition initiated by Richardson and maintained so well by his successors there has been an impressive flow of research papers from the department over the years, and many distinguished mathematicians have worked here. The publications have ranged widely over the two main branches, algebra and analysis, of pure mathematics as it was two or three decades ago. In the period after the second World War some of the older topics were worked out and others have grown in upon themselves, but in compensation there has been an astonishing proliferation of new branches of pure mathematics. This intensification and diversification has been reflected in the research output and interests of the department ; the range of subjects has become much wider. Richardson would applaud the multiplicity of research seminars currently being held in the department which he founded. My own main research preoccupation for the last few years has, however, been in a branch of mathematics which is not algebra or analysis and cannot really be called one of the exciting new proliferations. It is in fact quite an old subject which has been rather quiescent for perhaps as long as a century or so, but which is now enjoying a great resurgence of interest and development, particularly outside Great Britain.

It is a regrettable fact that I cannot hope personally to promote the study of this interesting and increasingly important branch of mathematics over any very substantial period in this college, but it is a duty and a pleasure to use the opportunity afforded me by an inaugural lecture to make known to colleagues in other departments, and any other interested persons, the attractiveness, significance and utility of the subject in which many of my research interests lie. If they have research problems of the sort to be indicated later in this lecture, it may be a comfort to them to know that the mathematicians are not completely indifferent and are groping for unifying principles which may one day help to solve their problems.

II

The subject is variously known as combinatorial analysis, or combinatorial theory, or merely as combinatorics. The word 'combinatorial' seems to have emerged as descriptive of this sort of mathematics in the seventeenth century. A work of Leibniz, written in 1666, when he was twenty years of age, is usually regarded as the genesis of the subject. This work, entitled 'Dissertatio de Arte Combinatoria', is a most extraordinary treatise. After a synopsis of its contents in which Leibniz promises applications of his combinatorial art to law, theology, grammar, logic, music, poetry, and the mixing of colours, among many other topics, he defines God as an incorporeal substance of infinite power, and with two further definitions (of substance and of infinite power), followed by one postulate and four axioms, he proves in twenty one steps that God exists, Q.E.D.

After this opening fanfare there follows a good deal of material which also cannot be considered wholly mathematical, but embedded in it is a table of the coefficients in the binomial expansion of $(1+x)^n$ up to $n=12$, and some properties of these coefficients. He also enumerates the fifteen non-empty subsets of a set of four elements A, B, C, D, namely the elements themselves, six pairs AB, AC, AD, BC, BD, CD which he calls com2nations, four triples ABC, BCD, CDA, DAB which he calls con3nations, and one con4nation ABCD. His quaint notation has not survived.

I think it must be admitted that the mathematical results given in this work of Leibniz are slight, and that many of the applications he writes about are trivial and rather fantastic. One must remember that he was a mere twenty years of age when this tract was published and, although this is not too young an age for mathematical genius to flower, he had not then been in contact with the modern mathematics of his period. He first met Huygens in Paris in 1672, six years after the publication of the combinatorial dissertation, and that is when his real mathematical education began. Although it is customary, but not necessarily correct, to regard pure mathematics as a young man's game, one does not have to regard the early efforts of every young mathematician as blinding inspirations, and indeed Leibniz subsequently referred to his 1666 dissertation as ' one which might have been written by a youth just out of the schools who was not yet conversant with the real sciences.'

Nevertheless, in spite of its evident immaturity, unnecessary complication and over-exuberant claims, the work had some deeper significance and was part of a long-term project which occupied Leibniz off and on for many years. It is quite impossible to give an account of this project here. One of the basic ideas was that of an ' alphabet of human thoughts '. These were indefinable concepts which he called ' first terms ', and from these are obtained derived concepts in much the same way as words are constructed from letters of an alphabet. He wrote of a series of classes ; the first class consisted of his ' first terms ' ; the second class consisted of pairs of his ' first terms ' ; the third class of triples of his ' first terms ', and so on. The construction of classes in this way, although a trivial notion in itself, has an important initial role in many combinatorial processes.

But the most imaginative and perhaps the most prophetic part of the dissertation is the fanciful range of applications to which Leibniz thought he could put his combinatorial art. It turns out that today, three hundred years later, combinatorics does occur naturally and inescapably in a bewildering variety of contexts, many of them seemingly far removed from any sort of basic mathematical structure of a more traditional kind. These contexts, many of them related to the essentially discontinuous nature of molecular and subatomic structures or discrete processes which arise naturally in many ways, are of course very different from those envisaged by Leibniz, but he seems to have had an instinctive feeling that the underlying principles of what he called his combinatorial art would have significance in many very dissimilar fields. In this he was right.

III

It is now time for me to attempt to give you some definition of the nature and range of the subject-matter of combinatorics. This is not altogether easy. In a very general way it is concerned with arrangements of objects, usually finite in number, into sets according to some prescribed restrictions or pattern. The objects themselves and the type of pattern and arrangement can be extremely varied in character. Sometimes the restrictions are such that it is not immediately clear whether the required arrangement, sometimes also called a configuration or structure, can actually exist.

To take a well known elementary example, suppose we have a chessboard with two squares removed, from opposite corners of the board. We are thus left with sixty two squares. We have thirty one rectangles, or dominoes, of such a size that each domino covers two squares of the chessboard. The combinatorial problem is to decide whether the board of sixty two squares can be covered by the thirty one dominoes. Only a moment's reflection is needed to see that such an arrangement of the dominoes is not possible, since the board has thirty two squares of one colour and thirty of the other, whereas the domino covering would require thirty one of each colour. It would be pleasant, but perhaps rather dull, if every existence problem in combinatorics was as evident as this.

When there is no doubt that a configuration does exist, the next combinatorial problem might be to find a construction, or an algorithm, to obtain it. Thus suppose we have a number of towns and a system of one-way roads such that it is possible to travel from any one of these towns to any other using the network of roads. One asks the question whether it is possible to traverse every road of the network once and only once in a single continuous circuit. Sufficient theory is known in connexion with this problem to tell us that such a circuit exists if and only if the number of roads of the network entering any town is the same as the number of roads of the network leaving it. But when this condition is satisfied, as it usually is for road systems, it still remains to find an actual circuit. Trial and error methods with a pencil and a diagram of the network will soon produce a solution for a small number of towns, but for a general situation of a large number of intersection points of a directed network one needs a more systematic procedure ; such an algorithm has been devised for this particular problem. It enables a circuit of the kind required to be traced through the network without having to retrace one's path at any stage.

In passing I should remark that the above problem of traversing each road once and once only is, for some peculiar reason, much easier than that of visiting each town in the network once and once only. In the latter case there is no known criterion to determine the existence of the required path.

When a configuration has no problems as to existence or construction it may well be that what we require to know is the number of essentially distinct configurations or re-arrangements which are possible. Thus if the problem is to cover a chessboard with thirty two dominoes, where each domino covers two squares, it is not at all difficult to find several ways of doing this by mere trial and error, but it is not easy to determine how many different ways there are. In fact there are 12,988,816 ways. In case you think that this result was obtained by a mathematician with nothing better to do than fritter his time away on rather futile chessboard problems, I should mention that it was published in the Physical Review of 1961 as an example of an enumeration process of great interest in statistical mechanics.

Another class of enumeration problem of interest to certain sorts of physicists is typified by the problem of non-self-intersecting random walks. One takes a generalised chessboard ; that is, one with a sufficiently large number of rows and columns. A random walk from any prescribed square is a sequence of steps, one square at a time, taken either along a row or along a column with

equal probability in any of the four directions. The problem is to enumerate the number of random walks of n steps which do not cross themselves. It has received much attention from physicists and others but it remains unsolved.

A broad class of problems in combinatorics is concerned with maximisation (or minimisation) of some numerical measure attached to an element or set of elements. The well-known problem of the travelling salesman is an example of this. Here we have a finite number of towns and we know the distance between any two of them. We are not concerned here with the nature of the network of roads connecting the towns, but merely with the distances. In effect what we have is a mileage chart such as appears in the handbooks of the motoring associations. The problem is to find the shortest route which will visit every town once only and then return to its starting point. A solution must exist since there is a finite number of routes visiting each town once only, and among these there will be one, or more, of minimum length. The construction of this shortest route is, however, a matter of great difficulty and a general algorithm is not known. An upper bound for the minimum number of steps in such an algorithm has been found, but this does not seem to be very helpful.

Many of these extremisation problems can be related in one way or another to the ideas centring around the theory of flows in networks ; for instance the flow of traffic between a number of towns connected by a network of roads of known maximum capacities. There are often interesting results that the maximum measure of some set of elements is equal to the minimum measure of another set which is in some sense dual or complementary to the first. There is quite a collection of these max-min theorems, and in this part of combinatorics there are signs of some basic unifying combinatorial theory. One of the most beautiful of these theorems is a celebrated result of R. P. Dilworth, published in 1950, from which as a very special example we can deduce the rather astonishing, but possibly quite useless, result that in a set of $mn+1$ white mice (or any other creatures with similar breeding propensities) there is either a sequence of $m+1$ mice each a descendant of the previous one in the sequence, or there is a subset of $n+1$ mice no one of which is a descendant of another member of the subset.

I am aware that I have not defined the nature and the extent of my subject very precisely, but I hope that by regarding it as the study of problems of existence, construction, enumeration and extremisation of configurations or arrangements possible under given restrictions or prescribed patterns, as instanced by the examples I have mentioned, you will have a good general understanding of its essentially discrete nature and its extremely wide range of applicability.

Its problems are nearly always easy to propound, usually sufficiently so as to be intelligible to the man in the street, but their solutions, when they exist and are not trivial, can be excruciatingly difficult to obtain. Like the problems of arranging dominoes on chessboards which I have mentioned, they often seem far removed from any sort of practical utility to the extent of seeming entirely frivolous, but this is deceptive. Applications are very often near at hand. A short note of mine on a combinatorial topic a few years ago turned out to have some relevance to the enumeration of certain genetical types and I had requests for offprints from research workers in animal science, zoology, entomology, psychology, endocrinology, mammalian genetics, communicable diseases, dairy and poultry science, animal husbandry, and health and welfare from various parts of the world, but rather strangely not from Great Britain. I must admit I was rather pleased with this show of interest, for although the traditional toast is 'to Pure Mathematics, may it never be of the slightest use to anyone', and although I agree with the inner truth of this in that we study the subject to satisfy a compelling intellectual curiosity and not necessarily for any other reason whatsoever, I find it

pleasing when the intellectual pearls of the pure mathematician are found to fit into worthy settings in other sciences.

IV

In order to convey to you some of the peculiar flavour and fascination of combinatorics I would now like to relate the famous story of the thirty six officers. It concerns a conjecture which had to wait one hundred and seventy seven years for its resolution. I start the story not from the battlefield or the barrack square, but from the farmyard. Consider the problem of the farmer who wishes to compare the yields of four varieties A, B, C, D of wheat. A crude method of comparison would be for him to divide his field into four plots of equal area, each growing one variety of wheat. This would not be very satisfactory if the soil or growing conditions were of uneven quality in different parts of the field. There can be no certain way of eliminating such inequalities, but their effects are almost certainly diminished considerably if we grow each variety in four separate plots instead of one, the four plots being dispersed in different parts of the field. The design of such an experiment could be shown diagrammatically by a square

B	A	D	C
D	C	B	A
C	D	A	B
A	B	C	D

in which each variety occurs once in each row and once in each column, thus diminishing the effects of any systematic variation of soil quality or other relevant conditions along the rows or along the columns. A square of this sort with the property that each symbol occurs once in each row and once in each column is known as a Latin square.

Our farmer may have another problem on his mind. He has four fertilizers α, β, γ, δ and he would like to test their relative effectiveness on his wheat crop. The cunning way of doing this is to distribute α, β, γ, δ among the sixteen plots so that each fertilizer appears once in each row and once in each column and further so that the sixteen pairs of seed and fertilizer are all different. One way in which this could be done is shown in the following diagram :

Bγ	Aδ	Dα	Cβ
Dβ	Cα	Bδ	Aγ
Cδ	Dγ	Aβ	Bα
Aα	Bβ	Cγ	Dδ

From the experiment designed in this way we can measure the relative effectiveness of α, β, γ, δ each acting on four plots containing the four varieties A, B, C, D, at the same time diminishing the effects of any variations over the field as a whole.

For those who have no interest in this exercise in plant-breeding, I mention in passing that if we replace A, B, C, D by Ace, King, Queen, Jack, and α, β, γ, δ by Clubs, Diamonds, Hearts, Spades, the above square gives a solution of the ancient problem of placing the sixteen court cards in a square array so that no row, no column and neither diagonal has more than one card from each suit and one of each rank.

Let us now distil the mathematical essence of these exercises in wheat-growing and card-placing. What we have done is to superimpose a Latin square based on α, β, γ, δ on a Latin square based on A, B, C, D so that the resulting pairs of one Greek and one Latin letter are all different. In general we have to superimpose an nxn Latin square on another nxn Latin square so that the n^2 pairs of elements are all different. We see from our example above that this can be done when n is four, but a question which arises in the mathematical mind is whether this can be done for every value of n. When n is two it cannot be done, since the only Latin squares we could superimpose would be

A	B		α	β
		and		
B	A		β	α

which give

Aα	Bβ
Bβ	Aα

which has only two distinct pairs and not four as required.

When n is three, it is possible since

A	B	C		α	β	γ
C	A	B	and	β	γ	α
B	C	A		γ	α	β

are Latin squares which when superimposed give

Aα	Bβ	Cγ
Cβ	Aγ	Bα
Bγ	Cα	Aβ

which has the requisite nine distinct pairs. It is also possible when n is five, and I expect most non-mathematicians at this stage would be prepared to assume that when n is greater than two it will be possible, with some trial and error and perhaps a good deal of patience, to superimpose two nxn Latin squares so that the resulting n^2 pairs of symbols are all different.

This is where I must introduce the great Swiss mathematician Leonhard Euler, (1707-83), and his thirty six officers. In 1782, as a mathematical diversion, Euler considered the problem of arranging thirty six officers in a square array according to a particular pattern. There were six officers from each of six different regiments, and each officer had one of six different ranks. No two officers from any one regiment were of the same rank. The problem was to arrange them so that each row and each column contained exactly one officer of each regiment and one officer of each rank. This is nothing more than the problem of superimposing two 6x6 Latin squares to give thirty six different pairs of rank and regiment. If you dislike the military context you can turn the swords into ploughshares and we are back on the farm with a problem of six varieties of wheat and six fertilizers.

The problem baffled Euler, which seems quite remarkable since he was a mathematician of very high order and was able to prove the powerful results that the general problem of n^2 officers of n different ranks from each of n different regiments could always be solved when n is an odd number or when n is a number divisible by four. He was unable to prove anything for the numbers outside these two categories, namely the numbers of the infinite arithmetical series 6, 10, 14, 18, 22, 26, . . ., and in 1782 he made his famous conjecture that the problem of n^2 officers had no solution when n had any of these values.

In 1901 his conjecture was proved to be correct when n is six by an exhaustive listing of all 6x6 Latin squares and showing that no two of them when superimposed would give thirty six different pairs. The next case, when n is ten, was subjected to computer investigation in 1959, but the computer produced no solution after something like five days of running time and so it seemed as if the Euler conjecture was also true when n is ten. It is only fair to mention however that 1959 computers were much slower than those of today. But at about the same time, 1958-59, the conjecture was completely settled by means other than computer investigation. E.T. Parker, R.C. Bose, and S.S. Shrikhande actually constructed solutions, first when n is twenty two and then when n is ten, and crowned their work by establishing the existence of solutions for the whole infinite arithmetical series, 10, 14, 18, 22, 26, ... Thus Euler's conjecture of 1782 was proved to be correct when n is six in 1901, and incorrect for the other values of n in 1959. It seems strange, and indeed almost mystical, that apart from the rather trivial case when n is two, there should be just the one number, six, for which this problem has no solution.

Problems concerning configurations of Latin squares and similar structures are by no means merely mathematical recreations. The ideas implicit in these problems form part of an important branch of combinatorial theory dealing with block designs and the design of experiments. Much work is currently in progress in this field.

V

If Euler's conjecture and its subsequent history show the fascination and unexpected solution of a combinatorial problem, my next topic shows above all else how exasperating a combinatorial conjecture can be. This is the notorious four-colour map conjecture. It seems to have arisen about 1850, when a certain graduate student, Francis Guthrie, in London, was drawing a map of England and observed that he needed only four colours to distinguish the counties in the customary geographical manner of giving each county a colour such that two counties with part of a boundary in common have different colours. In the general situation we have a finite plane area covered by a finite number of non-overlapping regions, no region consisting of two or more disconnected pieces. We wish to colour the regions so that two regions with a common boundary other than a single point have different colours. The conjecture, dating from about 1853, is that four colours are sufficient for colouring any such map. Although the conjecture refers to a map on a plane, there is no essential difference between a map on a sphere and a map on a plane in this context.

There is a two-colour map theorem which proves that if the surface of a sphere is divided into regions then the regions can be coloured with two colours in the way required if and only if the number of boundaries at each vertex, that is a meeting place of boundaries, is even.

There is as yet no general criterion for deciding when a map can be coloured with three colours only, but there are some partial results. Thus if there are at least five regions on the sphere and each region has a common boundary with three others then three colours are sufficient.

What was believed to be a proof that the four-colour conjecture holds was published in 1879, but eleven years later an error was found in this proof. The number of incorrect proofs put forward since then must run into hundreds ; sometimes the flaw is fairly obvious, but occasionally it is subtle and needs a careful unravelling of the proof to find it. One proof was demolished in a very recent issue of Mathematical Reviews. It is rumoured that another proof has recently been produced in the United States, but it has not yet been published and has not yet come through the very critical examination that any such proof must necessarily undergo. As this supposed proof has not yet been found to be correct beyond doubt, the situation is that the conjecture remains unresolved. It would be disproved if a map necessarily requiring five colours was discovered, but a direct search for such a conjecture-resolving map is not recommended. It was known in 1946 that such a map would need thirty six regions or more, and like everything else this number has gone up and is now at least forty. To test the immense number of topologically different maps with forty one regions to see if one of them cannot be coloured with four colours is an impossible task, with computers or without.

The situation is particularly exasperating in two respects. Firstly, it has been known since 1890 that five colours are sufficient for any map on the plane or sphere. Secondly, conjectures analogous to the four-colour conjecture for planar maps have been resolved for surfaces apparently much more complicated than the plane or sphere. For example seven colours are sufficient for any map on a torus, and six are sufficient for any map on the surface of a Klein bottle, which is a peculiar vessel, thought up by topologists, for which it is not possible to distinguish the inside from the outside ; further, there are maps on these surfaces which require seven and six colours respectively.

It is usual to deal with these map-colouring problems by taking a point within each region and, whenever two regions have a common boundary, joining the points corresponding to the two regions. In this way the map is replaced by a configuration of points and lines, and we have a combinatorial situation in which we have to colour the points so that the points at the end of each line have different colours. A configuration of points with lines

joining some or all of them is called a graph, and a good deal of combinatorial theory has grown up around graphs. It forms a major part of combinatorial theory. The four-colour map conjecture is transformable into several equivalent, and very interesting, conjectures in graph theory, but they all remain unresolved. There have been indications recently that some newer concepts rather more subtle than that of a graph may have relevance here, and they raise a wan hope that the conjecture may be resolved this century and mathematicians' minds put to rest on this problem. It is almost certain that the resolution of the conjecture will reveal some new and deeper results of graph theory in particular and of combinatorial theory in general. The effect on cartography will probably be absolutely nil.

VI

I will conclude my lecture with some sober reflexions on Leibniz' vision of a combinatorial art of wide applicability. It is perhaps more realistic to think of the multiplicity of disciplines in which combinatorial problems arise and how far we seem to be from devising principles of some generality which will lead to their solution, rather than to presuppose the existence of a compact body of mathematical knowledge waiting to be discovered which has the answers to all combinatorial problems. Awkward combinatorial problems arise in molecular biology, genetics, statistical mechanics, network theory, operational research, and mathematical economics, among many other sciences, and we certainly cannot solve all of them, nor in many of these have we as yet any general principles of combinatorial theory to apply to them. The problems are so varied that it is unlikely that they will all yield to just a few unifying general principles. Although accumulated knowledge is beginning to take systematised shape in some areas of combinatorial theory, for example, transversal theory, matroids, block designs, network flow, certain kinds of enumeration, and graphs, much remains outside these areas in a state of outer darkness illuminated here and there by some flash of individual genius. There is much to be explored, and in many directions, before the greatest truths of combinatorial theory are revealed. Those who seek to wrest these truths from the deep fastnesses in which they lie will need devilish ingenuity, craftiness, a good deal of patience, and probably genius of a rare kind. Perhaps one day we will be able to put the right questions in the right forms and get the right answers. In the meantime think kindly of the pure mathematicians, however useless, abstract, irrelevant, and new-fangled their lines of approach may appear. One of the lessons of scientific history is that what the pure mathematicians conjure from their intellectual flights of fancy quite often crystallises into tangible material of most unexpected usefulness and practical importance.